SOME DETERMINANT EXPANSIONS.*

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§ 1. There is a recent paper by Sir Thomas Muir† which presents an important general theorem upon the expansion of a determinant. Muir states the theorem summarily as follows:

A determinant can be expressed in terms of minors drawn from four mutually exclusive arrays, two of which are coaxial and complementary to one another.

The discussion leading up to this statement involves bordered determinants. But without reference to such determinants, by following a line of thought suggested by matter contained in §§ 11 and 12 of Muir's paper, it will be found possible not only to prove the theorem in a very simple manner but also to obtain several progressively broader results.

We need first, however, to state the theorem more specifically and in such a manner as to prepare for its extension. With respect to the four arrays it is to be noted that they may be marked out by drawing two lines, one horizontal and the other vertical, across the matrix of the determinant Δ, intersecting on the main diagonal line between two elements thereof; an arrangement which we shall denote as follows:

$$\|\Delta\| \equiv \begin{Vmatrix} A & B \\ C & D \end{Vmatrix};$$

$$A \equiv \begin{Vmatrix} a_{11} & \cdots & a_{1p} \\ \cdot & \cdot & \cdot \\ a_{p1} & \cdots & a_{pp} \end{Vmatrix}, \qquad B \equiv \begin{Vmatrix} b_{11} & \cdots & b_{1q} \\ \cdot & \cdot & \cdot \\ b_{p1} & \cdots & b_{pq} \end{Vmatrix},$$

$$C \equiv \begin{Vmatrix} c_{11} & \cdots & c_{1p} \\ \cdot & \cdot & \cdot \\ c_{q1} & \cdots & c_{qp} \end{Vmatrix}, \qquad D \equiv \begin{Vmatrix} d_{11} & \cdots & d_{1q} \\ \cdot & \cdot & \cdot \\ d_{q1} & \cdots & d_{qq} \end{Vmatrix}, \qquad p + q = n.$$

Further we must particularize with respect to the words "expressed in terms of minors drawn from" the arrays. It is well known that if a set of minors of a determinant Δ, such that their row numbers together are the row numbers of Δ and their column numbers the column numbers of Δ, be taken as the factors of a product to which is prefixed the sign of that term of Δ whose elements are the elements of the main diagonal terms of the minors, this product is identical with the sum of a certain number

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† Note on the representation of the expansion of a bordered determinant, by Sir Thomas Muir, LL.D., *Mess. Math.*, No. 566, Vol. xlviii., June, 1918.

of terms of Δ. As elsewhere,* we shall call any two minors of a determinant, which are susceptible of entering into such a set, *conjunctive* minors, and the whole a set of *perjunctive* minors or a *perjunct*; it is a *signed* perjunct if the specified sign is prefixed. When all the minors are of the first order or simply elements of Δ, we have a *transversal* of Δ; if signed, a *term.* The meaning of the phrase in the theorem then is that every possible signed perjunct is to be formed whose minors are four in number and lie one in each of the four arrays. It is understood that any one or more of these minors may be of order zero, with the value 1. And throughout this paper a perjunct will be understood to admit minors of order zero. Let a minor lying wholly in A be called an a-minor, and so for B, C, and D.

We are now ready to restate Muir's theorem as

THEOREM A. *If the matrix of any determinant Δ be partitioned, by a horizontal and a vertical line intersecting on the main diagonal line, into four arrays, A, B, C, and D, then Δ can be expanded as the sum of all signed perjuncts composed of an a-minor, a b-minor, a c-minor, and a d-minor.*

§ 2. In the proof of this theorem which we shall now give, and in further proofs in this paper, our line of thought concerns the individual terms of the determinant to be expanded, in their relation to the specified arrays into which the matrix of the determinant is divided.

Consider then any term of Δ. Separate its elements into those lying in A, those lying in B, those lying in C, and those lying in D. The four groups of elements determine by their row and column numbers four minors lying respectively in the four arrays and forming a perjunct which is evidently the only perjunct of four minors lying in the four arrays which contains this term.

Thus the sum of perjuncts specified in the theorem contains nothing but terms of Δ and contains every term once and only once. It is therefore an expansion of Δ.

§ 3. We are immediately led to give the theorem additional breadth by removing the condition that the horizontal and vertical lines must intersect on the main diagonal line, for the proof does not hang upon that condition; and we have

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§ 4. To illustrate this expansion let us take a determinant of order 7, partitioned thus:

$$\Delta \equiv \begin{vmatrix} \| a_{35} \| & \| b_{32} \| \\ \| c_{45} \| & \| d_{42} \| \end{vmatrix}.$$

* P-way determinants, with an application to transvectants, AMERICAN JOURNAL OF MATHEMATICS, Vol. XL, No. 3, July, 1918, p. 242. Cited herein as P-way dets.

(i) Take an a-minor of order 3, a d-minor of order 2, and the b-minor and c-minor determined thereby, the b-minor being of course of order zero while the c-minor is of order 2; example, $\underset{123}{a_{123}}\underset{45}{c_{12}}\underset{12}{d_{34}}$. (ii) Take an a-minor of order 2, a d-minor of order 1 (an element), and the b-element and c-minor of order 3 determined thereby; example, $-\underset{12}{a_{12}}\underset{1}{b_3}\underset{345}{c_{123}}\underset{2}{d_4}$. (iii) Finally, take an a-element (the d-minor now being of order zero), and the b-minor of order 2 and c-minor of order 4 determined by the a-element; example, $\underset{1}{a_1}\underset{12}{b_{23}}\underset{2345}{c_{1234}}$. As a check, we may count up in the result the terms of Δ: $\binom{5}{3}\binom{4}{2}3\,!\,2\,!\,2\,! + \binom{5}{2}\binom{3}{2}\binom{4}{1}\binom{2}{1}2\,!\,3\,! + \binom{5}{1}\binom{3}{1}2\,!\,4\,! = 7\,!$.

This procedure is applicable generally. We start by forming all possible perjuncts consisting of one of the largest a-minors and one of the largest d-minors, together with the b-minor and c-minor determined thereby; next we form all possible perjuncts with the a-minor and d-minor one less in order, and the b-minor and c-minor one greater; and so we continue, until the a-minor or the d-minor becomes of order zero.

§ 5. The next generalization consists in removing altogether the restrictions on the manner of partitioning the matrix of Δ into rectangular arrays. Let us call a rectangular array which is a part of the matrix of Δ a *panel* of Δ. Panels may be of any number and each may be of any dimensions so long as all fit together into the square matrix. With slight and obvious changes the former proof covers this more inclusive case, and we have

THEOREM 2. *If the matrix of a determinant Δ be partitioned into panels in any manner, then Δ can be expanded as the sum of all signed perjuncts composed of one minor from each panel.*

§ 6. A theorem was announced by Albeggiani in 1875 in a paper entitled *Sviluppo di un determinante ad elementi polinomi*,* which interests us here for three reasons. First, it can be proved in the manner of § 2 with directness and brevity. Secondly, it can be utilized to establish Theorem 2. And thirdly, it can be generalized from two dimensions to three or more.

As Albeggiani himself pointed out, this theorem applies to any determinant whatever, for polynomial elements can be made out of monomial elements *ad libitum*, either by breaking up the monomial elements or by annexing zero terms. Consider then the general determinant $\Delta = |a_{1n}|$, and put

$$a_{ij} = h_{ij}^{(1)} + h_{ij}^{(2)} + \cdots + h_{ij}^{(r)}.$$

Set up the r determinants

$$\Delta^{(k)} \equiv \begin{vmatrix} h_{11}^{(k)} & \cdots & h_{1n}^{(k)} \\ \cdot & \cdot & \cdot \\ h_{n1}^{(k)} & \cdots & h_{nn}^{(k)} \end{vmatrix}, \qquad k = 1, 2, \cdots, r.$$

* *Giorn. di Batt.*, Vol. 13, p. 1.

Form what we may call a *mixed perjunct* by taking one minor from $\Delta^{(1)}$, a second minor, conjunctive in position, from $\Delta^{(2)}$, and so on to $\Delta^{(r)}$, and prefix the sign determined precisely as it would be determined if all the minors came from one determinant. Then we may state Albeggiani's theorem as follows:

If Δ be any determinant, the sum of all the signed mixed perjuncts from r determinants so formed that the sum of their matrices is the matrix of Δ, is an expansion of Δ.

To prove the theorem, consider any term of Δ. This a-term yields r^n monomials each the product of n h's, which may be called h-terms of Δ. Now obviously we can think of expanding Δ directly into its h-terms without first forming the a-terms. And from that point of view it is clear that any given h-term is to be found in one and only one mixed perjunct. For, separate the elements of this h-term into the $h^{(1)}$'s, the $h^{(2)}$'s, and so on. These groups determine just one mixed perjunct containing this term; and therefore, as the perjuncts contain nothing but h-terms of Δ, we have an expansion of Δ.

§ 7. In order to prove Theorem 2 by means of Albeggiani's theorem, we form $\Delta^{(1)}, \Delta^{(2)}, \cdots, \Delta^{(r)}$ by writing into r blank matrices the r panels of Δ, each in its proper place, and then filling up each matrix with zeros. All minors of $\Delta^{(1)}$ vanish except those lying in the first panel, all minors of $\Delta^{(2)}$ except those lying in the second panel, and so on. The mixed perjuncts which survive are identical with the perjuncts of Δ specified in Theorem 2.

§ 8. Let us next extend Albeggiani's theorem to cubic or 3-way determinants, preparatory to an extension to p-way determinants. Let*

$$\Delta = |a_{\breve{\alpha}\frown{\beta}\frown{\gamma}}|_n; \qquad a_{\alpha\beta\gamma} = h^{(1)}_{\alpha\beta\gamma} + h^{(2)}_{\alpha\beta\gamma} + \cdots + h^{(r)}_{\alpha\beta\gamma}.$$

Set up the r determinants

$$\Delta^{(k)} = |h^{(k)}_{\breve{\alpha}\frown{\beta}\frown{\gamma}}|_n; \qquad k = 1, 2, \cdots, r.$$

Then we have

THEOREM 3. *If Δ be any 3-way determinant, the sum of all the signed mixed perjuncts from r determinants of the same signancy as Δ, the sum of whose matrices is the matrix of Δ, is an expansion of Δ.*

The proof follows that of § 6 very closely, the introduction of the nonsignant third index giving no trouble.

Defining a *block* as a 3-way rectangular matrix forming a part of the matrix of Δ, we have the

COROLLARY. *If the matrix of a 3-way determinant Δ be partitioned into blocks in any manner, then Δ can be expanded as the sum of all signed perjuncts composed of one minor from each block.*

* For the notation, see *P*-way dets., §§ 5, 6.

In particular, the blocks may be formed by three mutually perpendicular planes passed through the matrix. The types of perjuncts become here much more numerous than in the case of a 2-way determinant under Theorem 1. In any speical determinant there may be blocks the character of whose elements will simplify the application of the Corollary.

§ 9. Finally, consider the general p-way determinant

$$\Delta = |a_{\alpha\beta\cdots\kappa\lambda}|_n^{(p)},$$

in which any or all of the indices may be signant or nonsignant. Put

$$a_{\alpha\beta\cdots\kappa\lambda} = h^{(1)}_{\alpha\beta\cdots\kappa\lambda} + h^{(2)}_{\alpha\beta\cdots\kappa\lambda} + \cdots + h^{(r)}_{\alpha\beta\cdots\kappa\lambda},$$

and form r determinants of the same signancy as Δ:

$$\Delta^{(k)} = |h^{(k)}_{\alpha\beta\cdots\kappa\lambda}|_n^{(p)}, \qquad k = 1, 2, \cdots, r.$$

THEOREM 4. *If Δ be any p-way determinant, the sum of all the signed mixed perjuncts from r determinants of the same signancy as Δ, the sum of whose matrices is the matrix of Δ, is an expansion of Δ.*

Proof. First, to show that a signed perjunct consists of a certain number of terms of Δ. That the perjunct consists of transversals of Δ, is clear. It is now to the correspondence of signs that we must look. And it will be perceived that this point is really settled by the known correspondence in the case of a 2-way determinant. For, the argument in that case considers, first, row numbers, next, column numbers, treating both sets in the same way and combining the results. Here we have simply to apply the same argument to each signant index in turn, and to combine the results by taking the product of the signs of the signant ranges.

Secondly, to find any given h-term of Δ in one and only one mixed perjunct. We group the $h^{(1)}$'s, the $h^{(2)}$'s, and so on. Previous reasoning is here followed and the result readily comes, completing the proof.

Extend the definition of a *block* to p dimensions: it is to consist of all those elements for which α has a value found among a fixed set of values $\alpha_1, \alpha_2, \cdots, \alpha_{b_1}$; β, a value found among a set of values $\beta_1, \beta_2, \cdots, \beta_{b_2}$; and so on. The locant of the block is thus

$$\left\{\begin{matrix} \alpha_1\alpha_2 \cdots \alpha_{b_1} \\ \beta_1\beta_2 \cdots \beta_{b_2} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \lambda_1\lambda_2 \cdots \lambda_{b_p} \end{matrix}\right\}.$$

We shall then evidently have, under Theorem 4, the

COROLLARY. *If the matrix of a p-way determinant Δ be partitioned into blocks in any manner, then Δ can be expanded as the sum of all signed perjuncts composed of one minor from each block.*

§ 10. It is important to note that all of the foregoing results apply to permanents as well as to determinants, since the reasoning in no case depends—as does, for instance, the reasoning which establishes the multiplication theorem—upon the vanishing of certain aggregates of terms.

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